ONE-DIMENSIONAL DISPERSION OF A FINITE MASS OF PARTICLES OF LIKE CHARGE CONCENTRATED IN A PLANE

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With the aid of kinetic theory, a solution is obtained for the problem of one-dimensional dispersion into vacuum of charged particles that at an initial moment of time are concentrated in the plane x = 0. The problem is solved in the approximation of collisionless equations with allowance for a self-consistent electric field.

A unique asymptotic expression for one-dimensional dispersion is obtained. The relation between the dispersion problem of a charged gas layer and expansion from a point is demonstrated.

§1. At time t = 0 let N particles of like charge be concentrated in a plane x = 0; these particles possess a given velocity distribution function $f = f_0(u)\delta(x)$, where $\delta(x)$ is the Dirac delta function normalized by the condition $\int \delta(x)dx = 1$, in the case where the domain of integration includes the origin of coordinates.*

The specific form of the $f_0(u)$ function should be determined from considerations associated with the actual physical problem which, by way of abstraction, leads to the problem of expansion from a point. The motion of charged particles which, at t > 0, begin to disperse freely into vacuum will be described by a scheme of collisionless particle motion in a self-consistent electric field.

Vlasov's system of equations, conventionally used to study such motions, has the form

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + \frac{\varepsilon}{m}E(t, x)\frac{\partial f}{\partial u} = 0,$$

$$\frac{\partial E}{\partial x} = 4\pi\varepsilon\int_{-\infty}^{+\infty} f(t, x, u)\,du \ , \qquad (1.1)$$

where f is the distribution function of the particles, E is the electric-field intensity, ε is the charge of a particle, m is the mass of a particle, u is the velocity of a particle, t is time, and x is a Cartesian coordinate. External electric and magnetic fields are absent. Beyond the region of dispersion under study, at a sufficiently large distance from the origin of the coordinates, charges of unlike sign may always be distributed symmetrically, because such a distribution does not affect particle motion. On reaching a region of opposite charge, the dispersing particles are neutralized.

The system of equations (1.1) is essentially a nonlinear one, so that it is difficult to obtain a solution of the given problem directly from this system of equations. In the following, we shall use a different approach to the solution of the problem, in which the solution obtained will be identical with the solution of the system of equations (1.1).

Let us examine the equation for the electric-field intensity

$$\frac{\partial E}{\partial x} = 4\pi\varepsilon\int_{-\infty}^{+\infty} f du \qquad \left(\varepsilon\int_{-\infty}^{+\infty} f du = \rho(t, x)\right). \qquad (1.2)$$

The quantity $\rho(t, x)$ is the density of the charge distributed along the x axis at time t. Because of the symmetry of the problem E(0, t) = 0, the equation (1.2) may be written in the form

$$E(t, x) = 4\pi q \qquad \left(q = \int_0^x \rho(t, x)\,dx\right). \qquad (1.3)$$

Hence, it may be seen that the magnitude of the electric-field intensity at point x is equal to the magnitude of the total charge q distributed over the interval [0, x], multiplied by 4π.

We shall now examine an arbitrary particle which at time t = 0 began to move from point x = 0 at velocity u_0.

The displacement of the particle over a small period of time Δt is defined by the formula $\Delta x = u_0\Delta t$; hence all particles, the velocities of which at t = 0 were greater than u_0, will outdistance the particle examined, while all particles with velocities less than u_0 will lag behind.

This particle order, depending on initial velocities, will remain unchanged.

Indeed, the particles situated ahead of the particle examined possess velocities higher than u_0, so that their motion will be accelerated by an electric field of higher intensity, since the total charge q behind them will be greater than the total charge behind the particle examined; they will, therefore, continue to outdistance the particles with an initial velocity u_0.

For the same reason, the particles with an initial velocity less than u_0 will not be able to overtake the particle examined.

This means that behind a particle with initial velocity u_0 will be a constant number of particles, so that the electric-field intensity to which this particle is exposed will also be a constant.

The equations of motion of the particle may be written in the following form:

$$m\,du/dt = \varepsilon E(u_0) \qquad (E(u_0) = \text{const}). \qquad (1.4)$$

Integration of the equation (1.4) with the initial condition $u = u_0$ at t = 0 yields

$$u = \frac{\varepsilon}{m}E(u_0)\,t + u_0\ . \qquad (1.5)$$

*With respect to one-dimensional motions, one should bear in mind that all quantities are referred to unit area of the plane normal to the x axis.

Since $u = dx/dt$, the equation (1.5) may be integrated with the initial condition $x = 0$ at $t = 0$

$$x = \frac{\varepsilon}{2m} E(u_0) t^2 + u_0 t. \quad (1.6)$$

We shall now determine the specific form of the relation between the quantity E and the initial velocity of the particle u_0.

The number of particles situated behind a particle that began its motion at the velocity u_0 we shall denote by m_0. The total charge of these particles is then $q = \varepsilon m_0$. Hence, using the relation (1.3), we may write

$$E(u_0) = 4\pi\varepsilon \int_0^{u_0} f_0(u)\,du \qquad \left(m_0 = \int_0^{u_0} f_0(u)\,du\right). \quad (1.7)$$

Substituting this expression into (1.6), we obtain the equation of motion of the particle

$$x = \frac{4\pi\varepsilon^2}{2m} t^2 \int_0^{u_0} f_0(u)\,du + u_0 t. \quad (1.8)$$

Substituting the expression (1.7) for $E(u_0)$ in the equality (1.5), we get

$$u = \frac{4\pi\varepsilon^2}{m} \int_0^{u_0} f_0(u)\,du\,t + u_0. \quad (1.9)$$

The equation (1.8) yields an implicit dependence of u_0 on x and t, hence the relation (1.9) in combination with (1.8) yields the velocity distribution in the flow, i.e., (1.8) and (1.9) make it possible to determine u as a function of x and t. In combination, (1.7) and (1.8) make it possible to determine E as a function of x and t.

We shall now calculate the probability density function of the particles.

The number of particles with the initial velocities in the range $(u_0, u_0 + \Delta u_0)$ is equal to $\Delta N = f_0(u_0)\Delta u_0$. The particles with the initial velocity u_0 will be situated at the moment t at the point

$$x = \frac{4\pi\varepsilon^2}{2m} t^2 \int_0^{u_0} f_0(u)\,du + u_0 t.$$

Particles with an initial velocity $u_0 + \Delta u_0$ will be at time t at the point

$$x_1 = \frac{4\pi\varepsilon^2}{2m} t^2 \int_0^{u_0+\Delta u_0} f_0(u)\,du + (u_0 + \Delta u_0)\,t.$$

Correspondingly, ΔN particles will occupy the intercept $\Delta x = x_1 - x$. Hence, the density of the number of particles n at the point x at time t may be calculated as the limit ratio of ΔN to Δx for Δu_0 tending to zero

$$n = \lim_{\Delta u_0 \to 0} \frac{\Delta N}{\Delta x} = \frac{f_0(u_0)}{2\pi\varepsilon^2 m^{-1} f_0(u_0)\,t^2 + t}. \quad (1.10)$$

The equations (1.10) and (1.8) yield the dependence of n on x and t. Thus, we have determined all the macroscopic dispersion characteristics. From the relation (1.9), it may be seen that at $t = \infty$, the velocity of the gas particles becomes infinitely large. The equation (1.10) indicates that at sufficiently large t, the particle density has the asymptotic form

$$n \approx \frac{2m}{4\pi\varepsilon^2} \frac{1}{t^2}$$

which is independent of the velocity distribution of the particles at the initial moment of time.

It should be noted that this asymptotic expression for the density is incorrect only in the case of a small number of particles with initial velocities in the proximity of the point $u_0 = \infty$, because $f_0(\infty) = 0$, so that in the expression (1.10) the first term in the denominator becomes larger than the second. The asymptotic form, therefore, cannot be extended up to the point $x = \infty$ for finite values of t. The asymptotic representation of the velocity has the form $u = 2x/t$.

As distinct from the dispersion of charged particles, the asymptotic form of the collisionless dispersion of neutral particles is defined by the form of the initial distribution function, so that the particle density decreases with time according to the $1/t$ law [1, 2].

It should be noted that the qualitative picture of the problem is obvious enough even without these quantitative considerations. Particles of like charge are repulsed, experiencing at the same time a constant acceleration, as a result of which the density decreases according to $n \sim 1/t^2$.

§2. We shall now examine the relation between the problem of particle dispersion from a point, which we have just considered, and the problem of the dispersion of a layer of charged gas particles. The expression for the distribution function f has the form

$$f = n_0 \left(\frac{m}{2\pi k T_0}\right)^{1/2} \exp\left(-\frac{m(2\psi + u^2)}{2kT_0}\right),$$

where T_0 is the temperature of the gas layer, and ψ is the electric potential determined from the equation

$$\frac{d^2\psi}{dx^2} + 4\pi\varepsilon n_0 \exp\left(-\frac{m\psi}{kT_0}\right) = 0 \qquad \left(\psi = \frac{d\psi}{dx} = 0, \quad x = 0\right).$$

The gas is assumed to be hot, i.e., the potential energy acquired along the path length s is postulated small compared to kT_0. This condition may be written in the form

$$\frac{\varepsilon E s}{kT_0} \ll 1 \quad \text{or} \quad \frac{4\pi\varepsilon q s}{kT_0} \ll 1. \quad (2.1)$$

The dispersion of a cold layer of charged gas ($T_0 = 0$) was examined in [3] on the assumption that there is no thermal-velocity distribution of particles in the gas layer.

Let the mean free path in the gas be greater than the dimensions of the s layer.

Then, after instantaneous removal of the bounding surfaces, the gas will disperse in a collisionless mode under the effect of a self-consistent electric field. The condition (2.1) means that during the initial phase of dispersion, the electric field does not yet substantially affect particle motion, so that the velocity distribution of the particles will be the same as in the case of collisionless dispersion of neutral particles. Subsequently, particle motion will be appreciably influenced by the electric field; however, the order in which they follow one another will not change.

At a sufficiently large distance from the origin of coordinates ($|x| > s$), the dispersion of the layer will occur in the same fashion as the dispersion from a point; here, the velocity distribution function of the gas particles at the point $x = 0$ should be taken in the form $f = f_0(u)\delta(x)$, where

$$f_0(u) = \int_{-1/2 s}^{+1/2 s} n_0 \left(\frac{m}{2\pi kT_0}\right)^{1/2} \exp\left(-\frac{m(2\psi + u^2)}{2kT_0}\right) dx = N_0 \left(\frac{m}{2\pi kT_0}\right)^{1/2} \exp\left(-\frac{mu^2}{2kT_0}\right).$$

Here, N_0 is the number of particles in a gas layer of thickness s. The macroscopic characteristics of the flow may be calculated from the expressions (1.8), (1.9), and (1.10).

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